

# The Magnetic Rayleigh-Taylor Instability in Astrophysical Disks

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## ABSTRACT

This is our first study of the magnetic Rayleigh-Taylor instability at the inner edge of an astrophysical disk around a central black hole. We derive the equations governing small-amplitude oscillations in general relativistic ideal magnetohydrodynamics and obtain a criterion for the onset of the instability. We suggest that static disk configurations where magnetic field is held by the disk material are unstable around a Schwarzschild black hole. On the other hand, we find that such configurations are stabilized by the spacetime rotation around a Kerr black hole. We obtain a crude estimate of the maximum amount of poloidal magnetic flux that can be accumulated around the center, and suggest that it is proportional to the black hole spin. Finally, we discuss the astrophysical implications of our result for the theoretical and observational estimations of the black hole jet power.

**Key words:** MHD; GR

## 1 INTRODUCTION

Astrophysical magnetic fields are believed to play a fundamental role in powering astrophysical energetic sources such as active galactic nuclei, X-ray binaries, and gamma-ray bursts. Extensive research over the last 4 decades has most convincingly shown that magnetic fields contribute to the extraction of rotational energy from astrophysical accretion disks and compact objects (neutron stars, black holes), and to the launching, collimation and acceleration of astrophysical jets. The electrostatically extracted power is proportional to the square of the total amount of open magnetic flux that threads the central spinning compact object. In the case of a spinning neutron star, the magnetic field originates in the stellar interior and is held in place by the highly conducting neutron star matter. In the case of a spinning black hole, however, the magnetic field is held in place by the surrounding disk of matter, and if the disk is removed, the magnetic field escapes the system at light crossing times.

The origin of the large scale astrophysical magnetic field held by the accretion disk around a spinning black hole is not clear. One school of thought suggests that the field is brought in from large scales by the accretion flow, and several numerical simulations are set up with a ‘reservoir’ of large scale poloidal magnetic flux at large distances (e.g.

Tchekhovskoy, Narayan & McKinney 2011). The main problem with this scenario is that astrophysical accretion disks are viscous, thus also diffusive, and therefore they can hardly advect any magnetic flux over so many orders of magnitude in radius (e.g. Lubow et al. 1994). To our understanding, the problem of how magnetic flux is brought in from large distances is still open (e.g. Lovelace et al. 2009; Kylafis et al. 2011). Another more promising astrophysically plausible scenario proposes that the magnetic field is generated around the inner edge of the accretion disk. This is the Cosmic Battery according to which, one polarity is advected inward and inundates the black hole horizon, whereas the return polarity diffuses outward through the surrounding disk (Contopoulos & Kazanas 1998; Contopoulos, Nathanail & Katsanikas 2015).

Whatever the origin of the magnetic field turns out to be, the common understanding is that the collected field is held in place by the ‘weight’ of the disk that keeps the magnetic field from escaping. According to this understanding, the growth of the field cannot continue beyond a so called equipartition limit  $B_{\text{eq}}$  where the magnetic field energy density either balances the accretion disk ram pressure, namely

$$\frac{B_{\text{eq}}^2}{8\pi} \sim M_{\text{disk}} \frac{v_K}{4\pi r^2}, \quad (1)$$

or balances the full weight of the inner disk, namely

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$$\frac{B_{\text{eq}}^2}{8\pi} \sim \frac{GMM_{\text{disk}}}{4\pi r^4} \quad (2)$$

(eq. 2 follows from eq. 1 for thick disks only). Here,  $M$  is the mass of the central black hole. When the magnetic field (or equivalently the total accumulated magnetic flux) reaches a value on the order of the above limits, accretion will be disrupted. Such configuration is termed Magnetically Arrested Disk (MAD; Igumenshchev 2008). Recent state-of-the-art numerical simulations have shown the MAD process in action. In axisymmetry (2D), when the accumulated magnetic field reaches the above maximum value, accretion stops. In realistic 3D accretion though, magnetic flux can escape the system in the azimuthal- $\phi$  direction as shown very clearly in the numerical simulations of e.g. Tchekhovskoy et al. (2011). The breaking of the axisymmetry by the azimuthal ‘bunching up’ of the field lines is precisely the *magnetic Rayleigh-Taylor instability*. And here rises the obvious question: how stable are MAD configurations against this instability?

In classical fluid motion, the Rayleigh-Taylor instability has been investigated by several authors in both hydrodynamics and magnetohydrodynamics (Chandrasekhar 1961; Kruskal and Schwarzschild 1954; an interesting presentation can be found in Boyd and Sanderson 1969). The aim of the present work is to determine more precisely the main parameters that characterize the onset of this important instability around astrophysical black holes. Numerical simulations yield the amount of magnetic flux that is effectively held around the central spinning black hole which, as we said, is a fundamental parameter that determines the efficiency of energy production in energetic astrophysical sources. We would like to be able to obtain the same result from first principles. This will allow us to determine whether an astrophysical black hole is active (implying that it generates jets that extract energy from its rotation) or inactive. Another very important future application of the present work would be to explain the various stages of a flaring X-ray binary where too, as shown in Kylafis et al. (2012) the main parameter that characterizes the evolution is the generation and destruction of the large scale magnetic flux accumulated around the black hole horizon.

The goal of this paper is to obtain the magnetic Rayleigh-Taylor stability criterion for an astrophysical disk with a central black hole. We were able to achieve our goal only in the simplified case of two static distributions of ideal magnetized plasma in contact with each other in the equatorial plane of the central black hole. We perturbed the contact interface in the radial and azimuthal direction and considered a particular form of velocity perturbations that allowed us to obtain a criterion for the stability of the interface. In the next section we develop our general relativistic formalism, and in § 3 we apply it to obtain the general stability criterion on the equatorial plane. In the next two sections we apply our results around non-rotating and rotating black hole respectively, and in the final section, we discuss the astrophysical implications of our work.

## 2 GENERAL RELATIVISTIC MHD IN 3+1 FORMALISM

We will follow the 3+1 (space+time) formalism of general relativistic magnetohydrodynamics (GRMHD) of Thorne &

Macdonald (1982). We introduce spatial magnetic and electric fields ( $\mathbf{B}$  and  $\mathbf{E}$  respectively) as measured by fiducial observers with 4-velocity  $U^\mu$ . In that formalism, Maxwell’s equations  $F_{;\beta}^{\alpha\beta} = 4\pi J^\alpha$ ,  $F_{[\alpha\beta;\gamma]} = 0$ , and  $J_{;\alpha}^\alpha = 0$  yield

$$\begin{aligned} \tilde{\nabla} \cdot \tilde{E} &= 4\pi\rho_e \\ \tilde{\nabla} \cdot \tilde{B} &= 0 \\ D_\tau \tilde{E} + \frac{2}{3}\theta \tilde{E} - \tilde{\sigma} \cdot \tilde{E} &= \frac{1}{\alpha} \tilde{\nabla} \times (\alpha \tilde{B}) - 4\pi \tilde{J} \\ D_\tau \tilde{B} + \frac{2}{3}\theta \tilde{B} - \tilde{\sigma} \cdot \tilde{B} &= -\frac{1}{\alpha} \tilde{\nabla} \times (\alpha \tilde{E}) \end{aligned} \quad (3)$$

with

$$D_\tau \rho_e + \theta \rho_e + \frac{1}{\alpha} \tilde{\nabla} \cdot (\alpha \tilde{J}) = 0. \quad (4)$$

Here,  $D_\tau M^\beta \equiv M^\beta_{;\mu} U^\mu - U^\beta a_\mu M^\mu$  is the Fermi derivative,  $\theta$  and  $\tilde{\sigma}$  are the expansion and shear of the spacetime metric respectively. The evolution of the magnetized fluid is characterized by the divergence of the total stress-energy tensor  $T^{\mu\nu} \equiv T_{\text{matter}}^{\mu\nu} + T_{\text{EM}}^{\mu\nu}$ , namely

$$T^{\mu\nu}_{;\nu} = 0, \quad (5)$$

which yields

$$\begin{aligned} D_\tau \varepsilon + \theta \varepsilon + \frac{1}{\alpha^2} \tilde{\nabla} \cdot (\alpha^2 \tilde{S}) + W^{jk} (\sigma_{jk} + \frac{1}{3} \theta \gamma_{jk}) &= -\tilde{J} \cdot \tilde{E} \\ D_\tau \tilde{S} + \frac{4}{3} \theta \tilde{S} + \tilde{\sigma} \cdot \tilde{S} + \varepsilon \tilde{a} + \frac{1}{\alpha} \tilde{\nabla} \cdot (\alpha \tilde{W}) &= \\ \rho_e \tilde{E} + \tilde{J} \times \tilde{B}. \end{aligned} \quad (6)$$

Here,

$$\begin{aligned} \varepsilon &\equiv T_{\text{matter}}^{\mu\nu} U_\mu U_\nu \\ S^\alpha &\equiv \gamma^\alpha_{\mu} T_{\text{matter}}^{\mu\nu} U_\nu \\ W^{\alpha\beta} &\equiv \gamma^\alpha_{\mu} T_{\text{matter}}^{\mu\nu} \gamma^\beta_{\nu} \\ \theta &\equiv U^\mu_{;\mu}, \quad a^\mu \equiv U^\mu_{;\nu} U^\nu, \\ \sigma_{ab} &\equiv \frac{1}{2} \gamma^\mu_a \gamma^\nu_b (U_{\mu;\nu} + U_{\nu;\mu}) - \frac{1}{3} \theta \gamma_{ab} \\ \tilde{L} \cdot \tilde{M} &= \gamma^{ij} L_i M_j, \quad (\tilde{L} \times \tilde{M})^j = \epsilon^{ijk} L_j M_k, \end{aligned} \quad (7)$$

and,  $\gamma^{\alpha\beta} = g^{\alpha\beta} + U^\alpha U^\beta$  is the projection tensor, and  $\alpha$  is the lapse function. Latin indices take values 1, 2, 3 and Greek ones 0, 1, 2, 3. Vectors and tensors with tildae are purely spatial. For an ideal fluid with density  $\rho$ , 3-velocity  $\tilde{v}$ , and pressure  $p$  we have

$$\begin{aligned} \Gamma &= (1 - \tilde{v}^2)^{-1/2}, \quad \varepsilon = \Gamma^2 (\rho + p \tilde{v}^2) \\ \tilde{S} &= (\rho + p) \Gamma^2 \tilde{v}, \quad \tilde{W} = (\rho + p) \Gamma^2 \tilde{v} \otimes \tilde{v} + p \tilde{\gamma}. \end{aligned} \quad (8)$$

We will also assume an equation of state  $p = p(\rho)$  from which we deduce the ‘speed of sound’

$$c_s \equiv \left( \frac{dp}{d\rho} \right)^{1/2}. \quad (9)$$

Finally, we will also assume ideal MHD conditions, namely

$$\tilde{E} = -\tilde{v} \times \tilde{B} \quad (10)$$

In order to investigate the Rayleigh-Taylor instability in an astrophysical context, we will now consider the special case of a Kerr space-time.

## 2.1 Kerr spacetime

In Boyer-Lindquist (BL) coordinates the Kerr metric reads

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \\ &= -(1 - \frac{2Mr}{\Sigma})dt^2 - \frac{4Mar\sin^2\theta}{\Sigma}dtd\phi \\ &\quad + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{A}{\Sigma}\sin^2\theta d\phi^2 \end{aligned} \quad (11)$$

where  $M$  is the mass of the black hole,  $a$  is the angular momentum per unit mass ( $0 \leq a \leq M$ ), and

$$\begin{aligned} \Delta &\equiv r^2 - 2Mr + a^2 \\ \Sigma &\equiv r^2 + a^2 \cos^2\theta \\ A &\equiv (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta \end{aligned} \quad (12)$$

(Cowling 1941). Notice that we work in geometrical units in which  $c = G = 1$ .

For our further study we need the components of the 4-velocity of fiducial observers, now identified as ZAMOs (Zero Angular Momentum Observers), namely

$$U^\mu = (\frac{1}{\alpha}, 0, 0, \frac{\omega}{\alpha}), \quad U_\mu = (-\alpha, 0, 0, 0) \quad (13)$$

where

$$\alpha = \sqrt{\frac{\Delta\Sigma}{A}}, \quad \omega = \frac{2Mar}{A} \quad (14)$$

In the Kerr spacetime (11) with 4-velocity  $U^\mu$  given by eq. (13), the expansion  $\theta$  vanishes, the shear  $\tilde{\sigma}$  has two non-zero components e.g. the  $\sigma^{13}$  and  $\sigma^{23}$  but  $\sigma_{\alpha\beta}\gamma^{\alpha\beta} = 0$ ; (see Thorne & Macdonald 1982, paper I, eq. (2.5)). The acceleration  $a^\mu$  is given by

$$\begin{aligned} a^\mu &= (0, \frac{-Ma^2 \cos^2\theta[(r^2 + a^2)^2 - 4Mr^3]}{\Sigma^2 A} \\ &\quad + \frac{Mr^2[(a^2 + r^2)^2 - 4Mr a^2]}{\Sigma^2 A} \\ &\quad , \frac{Mra^2(r^2 + a^2)\sin 2\theta}{\Sigma^2 A}, 0). \end{aligned} \quad (15)$$

$\gamma_{ij}$  is the spatial metric on the space-like hypersurface  $x^0 \equiv t = \text{const.}$ , with normal vector  $n_\alpha$

$$n_\alpha = (-\alpha, 0, 0, 0), \quad n^\alpha = \frac{1}{\alpha}(1, -\beta^1, -\beta^2, -\beta^3) \quad (16)$$

where  $\beta^i = \gamma^{ij}g_{0j}$ .

## 2.2 The perturbed MHD equations

We consider only small perturbations of physical quantities as

$$\begin{aligned} \rho(t, \tilde{r}) &= \rho_0(\tilde{r}) + \delta\rho(t, \tilde{r}) \\ \rho_e(t, \tilde{r}) &= \rho_{e0}(\tilde{r}) + \delta\rho_e(t, \tilde{r}) \\ v^i(t, \tilde{r}) &= v_0^i(\tilde{r}) + \delta v^i(t, \tilde{r}) \\ B^\mu(t, \tilde{r}) &= B_0^\mu(\tilde{r}) + \delta B^\mu(t, \tilde{r}) \\ E^\mu(t, \tilde{r}) &= E_0^\mu(\tilde{r}) + \delta E^\mu(t, \tilde{r}) \\ J^\mu(t, \tilde{r}) &= J_0^\mu(\tilde{r}) + \delta J^\mu(t, \tilde{r}) \end{aligned} \quad (17)$$

and keep only linear terms of the perturbations. In this case

$$\begin{aligned} v^2 &= \gamma_{ij}v^i v^j = \gamma_{ij}(v_0^i + \delta v^i)(v_0^j + \delta v^j) \\ &= \gamma_{ij}v_0^i v_0^j + 2\gamma_{ij}v_0^i \delta v^j \end{aligned}$$

$$\begin{aligned} &= \gamma_{rr}(v_0^r)^2 + \gamma_{\phi\phi}(v_0^\phi)^2 \\ &\quad + 2\gamma_{rr}v_0^r \delta v^r + 2\gamma_{\phi\phi}v_0^\phi \delta v^\phi \end{aligned} \quad (18)$$

and

$$\begin{aligned} \Gamma^2 &= \{1 - [\frac{\Sigma}{\Delta}(v_0^r)^2 + \frac{A}{\Sigma}\sin^2\theta(v_0^\phi)^2 \\ &\quad + 2\frac{\Sigma}{\Delta}v_0^r \delta v^r + 2\frac{A\sin^2\theta}{\Sigma}v_0^\phi \delta v^\phi]\}^{-1} \end{aligned} \quad (19)$$

In the Cowling approximation of a fixed Kerr spacetime,

$$\begin{aligned} \delta\Gamma^2 &= \frac{v_0^k \delta v_K}{(1-v^2)^2} + \frac{v_{0k} \delta v^k}{(1-v^2)^2} = 2\frac{\gamma_{kl}v_0^k \delta v^l}{(1-v^2)^2} \\ \delta\varepsilon &= 2(\rho + v^2 p) \frac{v_0^k \delta v_K}{(1-v^2)^2} \\ &\quad + \frac{1}{1-v^2}[\delta\rho + v_0^2 \delta p + 2pv_0^k \delta v_K] \\ \delta S^i &= \frac{v_0^i}{1-v^2}(\delta\rho + \delta p) + 2\frac{v_0^k \delta v_K}{(1-v^2)^2}(\rho + p)v_0^i \\ &\quad + \frac{\delta v^i}{1-v^2}(\rho + p) \\ \delta W^{ij} &= \frac{v_0^i v_0^j}{1-v^2}(\delta\rho + \delta p) + 2\frac{v_0^k \delta v_K}{(1-v^2)^2}(\rho + p)v_0^i v_0^j \\ &\quad + \frac{(\rho + p)}{1-v^2}(v_0^j \delta v^i + v_0^i \delta v^j) + \gamma^{ij} \delta p \end{aligned} \quad (20)$$

The first order perturbed MHD equations now become

$$\tilde{\nabla} \cdot \delta \tilde{E} = 4\pi \delta \rho_e, \quad (21)$$

$$\tilde{\nabla} \cdot \delta \tilde{B} = 0, \quad (22)$$

$$D_\tau \delta \tilde{E} = \tilde{\nabla} \times \delta \tilde{B} + \tilde{a} \times \delta \tilde{B} + \tilde{\sigma} \cdot \delta \tilde{E} - 4\pi \delta \tilde{J}, \quad (23)$$

$$D_\tau \delta \tilde{B} = -\tilde{\nabla} \times \delta \tilde{E} - \tilde{a} \times \delta \tilde{E} + \tilde{\sigma} \cdot \delta \tilde{B}, \quad (24)$$

$$D_\tau \delta \rho_e + \delta \tilde{J} \cdot \tilde{a} + \tilde{\nabla} \cdot \delta \tilde{J} = 0, \quad (25)$$

$$D_\tau \delta \rho + 2\delta \tilde{S} \cdot \tilde{a} + \tilde{\nabla} \cdot \delta \tilde{S} + \tilde{\sigma} \cdot \delta \tilde{W} = -\delta \tilde{J} \cdot \tilde{E}_0 - \tilde{J}_0 \cdot \delta \tilde{E}, \quad (26)$$

$$D_\tau \delta \tilde{S} + \tilde{a} \delta \rho + \delta \tilde{W} \cdot \tilde{a} + \tilde{\nabla} \cdot \delta \tilde{W} + \tilde{\sigma} \cdot \delta \tilde{S} \quad (27)$$

$$= (\delta \rho_e \tilde{E}_0 + \delta \tilde{J} \times \tilde{B}_0) + (\rho_{e0} \delta \tilde{E} + \tilde{J}_0 \times \delta \tilde{B}). \quad (28)$$

## 3 THE STATIC EQUATORIAL DISK

We will now restrict our analysis to the investigation of a static equatorial distribution of matter of thickness  $h \lesssim r$ . By ‘static’ we mean that the disk fluid is initially at rest with respect to ZAMOs, i.e. that  $v_0^i = 0$ . Our disk configuration only vaguely mimics astrophysical accretion disks. We acknowledge that neglecting the Keplerian disk rotation is an important simplification that we apply only to make some progress with the complex general relativistic formalism. However, our results may be physically relevant in Magnetically Arrested Disks in which rotation plays a secondary role (see discussion section).

The problem we have in mind is a distribution of matter consisting of two regions inside and outside some radius  $r_0$ . We will thus only consider perturbations in the immediate neighborhood of  $r_0$  of the form

$$\left. \begin{aligned} \delta\rho(t, r, \theta, \phi) &= \delta\rho(r) \\ \delta\rho_e(t, r, \theta, \phi) &= \delta\rho_e(r) \\ \delta v^i(t, r, \theta, \phi) &= \delta v^i(r) \\ \delta B^i(t, r, \theta, \phi) &= \delta B^i(r) \\ \delta E^i(t, r, \theta, \phi) &= \delta E^i(r) \\ \delta J^i(t, r, \theta, \phi) &= \delta J^i(r) \end{aligned} \right\} \cdot e^{nt+im\phi} \quad (29)$$

in the equatorial plane  $\theta = \pi/2$ , where  $m$  takes integer values 1, 2, 3... For simplicity, we will henceforth ignore the index ‘0’ from the zeroth order terms. In this case,

$$\begin{aligned} v^2 &= 0, \quad \Gamma^2 = 1, \quad \delta\Gamma^2 = 0 \\ \delta S^i &= (\rho + p)\delta v^i, \quad \delta W^{ij} = \gamma^{ij}\delta p, \quad \delta a^\mu = 0 \\ \delta\epsilon &= \delta\rho, \quad \sigma^{23} = 0, \quad E^i = 0, \quad i = r, \theta, \phi. \end{aligned} \quad (30)$$

We will also assume dipolar symmetry in the magnetic field, namely  $B^r(\pi - \theta) = -B^r(\theta)$ ,  $B^\theta(\pi - \theta) = B^\theta(\theta)$ , and  $B^\phi(\pi - \theta) = -B^\phi(\theta)$ . On the equatorial plane in particular,

$$B^i = (0, B^\theta, 0), \quad a^\theta = 0, \quad \Sigma = r^2, \quad (31)$$

and  $B_{r,\theta} \sim h_1 B_\theta/r$ ,  $B_{\phi,\theta} \sim h_2 B_\theta/r$ , with  $h_1, h_2 \approx \text{const.}$  In what follows, we will set for simplicity  $h_1 = h_2 = 0$ . The system of eqs. (3) & (6) now becomes

$$\begin{aligned} \rho_e &= 0 \\ B^\theta &= B^\theta(r) \\ J^r &= 0, \quad J^\theta = 0 \\ J^\phi &= \frac{\alpha}{4\pi r^2} [B_{\theta,r} + a_r B_\theta] \\ (\rho + p)a^r + \left(\frac{\Delta}{r^2}\right)p_{,r} &= -\frac{\alpha}{r^2} J_\phi B_\theta. \end{aligned} \quad (32)$$

The last equation in eqs. (32) may be written as

$$\left(p + \frac{B^2}{8\pi}\right)_{,r} = -a_r(\rho + p + \frac{B^2}{4\pi}) + \frac{B^2}{4\pi r} \quad (33)$$

To make further progress, we will assume one more simplification, namely

$$\tilde{\nabla} \cdot \delta\tilde{v} = 0. \quad (34)$$

Even under our present assumptions, the general system of first order equations is rather complicated. We thus decided to move our detailed calculations to the Appendix B. Eqs. (28) then becomes eq. (91) which, with the aid of eqs. (87) and (89), yields:

$$\begin{aligned} & \left[\left(\frac{A}{r^4}\right)(r^2\delta v^r)_{,r}(\rho + p + \frac{B^2}{4\pi})\right]_{,r} \\ & - a_r\left(\frac{A}{r^4}\right)(r^2\delta v^r)_{,r}(\rho + p + \frac{B^2}{4\pi}) \\ & = \left(\frac{m^2}{\Delta}\right)(\rho + p + \frac{B^2}{4\pi})(r^2\delta v^r) \\ & - \frac{m^2\alpha^2}{n^2 + m^2\omega^2}\left(\frac{r^2 a^r}{A}\right)\{[(1 - c_s^4)\rho]_{,r} \\ & - \left(\frac{1}{4\pi}\right)(1 + 3c_s^2)\left(\frac{r^2}{2}\right)[(B^\theta)^2]_{,r} \\ & - \left(\frac{3}{4\pi}\right)(1 + c_s^2)(a_r + \frac{2}{r})B^2\}\}(r^2\delta v^r). \end{aligned} \quad (35)$$

Eq. (35) is the general relativistic form of the ‘force balance’ equation, and is valid inside, outside, and across the interface  $r = r_0$  of two fluids in equilibrium on the equatorial plane.

In order to make further progress, we will assume that our physical quantities  $\rho$ ,  $p$ , and  $B$  are constant inside and

outside  $r_0$  and  $\delta\rho = \delta p = \delta B = 0$  (at least near  $r_0$ ), but may change discontinuously across  $r_0$ .  $\delta v^r$  and  $(p + \frac{B^2}{8\pi})$  are continuous<sup>1</sup> across the interface between the two fluids, but  $\rho$ ,  $B^2$  and  $(\delta v^r)_{,r}$  in general are not. For any physical quantity  $f$  discontinuous across  $r_0$  we define the jump

$$\mathcal{D}\{f\} \equiv f_{(2)} - f_{(1)}. \quad (36)$$

where

$$f_{(1)} \equiv f(r_0 - \epsilon), \quad \text{and} \quad f_{(2)} \equiv f(r_0 + \epsilon). \quad (37)$$

In that notation, eq. (35) yields

$$\begin{aligned} n^2 &= -m^2\omega^2 \\ &+ m^2\left(\frac{r^6 a^r}{A^2}\right)[(1 - c_s^4)\mathcal{D}\{\rho\} - (1 + 3c_s^2)\mathcal{D}\{\frac{B^2}{8\pi}\}] \\ &/[\mathcal{D}\{(\rho + p + \frac{B^2}{4\pi})(\delta v^r)_{,r}\}/\delta v^r] \end{aligned} \quad (38)$$

This is the most important equation in our analysis. It is the one that yields the general criterion for the onset of the magnetic Rayleigh-Taylor instability. The reader can see this directly by considering the simple un-magnetized Newtonian limit with  $\Delta = r^2$ ,  $A = r^4$ ,  $\omega = a = 0$ ,  $\alpha = 1$ , and  $p \ll \rho$ . In that limit, eq. (38) yields

$$n^2 = m^2 \frac{a^r \mathcal{D}\{\rho\}}{\mathcal{D}\{\rho(\delta v^r)_{,r}\}/\delta v^r}. \quad (39)$$

As we will see below, the above denominator is positive, and therefore, eq. (39) simply tells us that the Rayleigh-Taylor instability sets in (i.e.  $n^2 > 0$ ) when  $\mathcal{D}\{\rho\}$  has the opposite sign of that of gravitational acceleration  $g^r \equiv -a^r$ . The reader can easily convince him/herself that this indeed corresponds to a ‘heavy’ fluid above a ‘light’ one (like water over oil). This is reassuring enough for us to proceed with our investigation. Notice that  $c_s$  is discontinuous across  $r_0$ , and therefore, the terms involving  $c_s$  in eq. (38) simply imply average values across the discontinuity (i.e.  $c_s^2 \equiv ((c_s^2)_{(1)}^2 + (c_s^2)_{(2)}^2)/2$  and  $c_s^4 \equiv ((c_s^4)_{(1)} + (c_s^4)_{(2)})/2$ ).

The last missing piece is the calculation of the discontinuity of  $(\delta v^r)_{,r}$  across  $r_0$ . This may be obtained by solving eq. (35) inside and outside  $r_0$  where  $\rho$  and  $B^\theta$  are taken to be constant. Eq. (35) may be rewritten as

$$(\delta v^r)_{,rr} + P(r)(\delta v^r)_{,r} + Q(r)\delta v^r = 0 \quad (40)$$

where

$$\begin{aligned} P(r) &\equiv \frac{2}{r} + \frac{2}{A}(r^3 - a^2 r) - \frac{M}{r(r - 2M)} \\ Q(r) &= -\frac{2}{r^2} - \frac{2M}{r^2(r - 2M)} + \frac{4(r^3 - a^2 M)}{rA} \\ &\quad - \frac{m^2 r^4}{A\Delta} \\ &\quad - \left(\frac{\lambda_2}{n^2 + m^2\omega^2}\right)\left[\frac{M^2 r^2 \Delta}{A^2(r - 2M)^2}\right] \end{aligned} \quad (41)$$

where

$$\lambda_2 \equiv m^2 \frac{\frac{3B^2}{4\pi}(1 + c_s^2)}{\rho + p + \frac{B^2}{4\pi}} \quad (42)$$

<sup>1</sup> The continuity of  $\delta v^r$  is obvious. The continuity of  $(p + \frac{B^2}{8\pi})$  derives from the  $r$ -derivative terms in eq. (33).

In eq. (40) setting

$$\begin{aligned}\delta v^r(r) &= z(r) \exp\left[-\frac{1}{2} \int^r P(t) dt\right] \\ &= z(r) \sqrt{\frac{\alpha}{A}}\end{aligned}\quad (43)$$

where

$$I = \frac{1}{2} \int^r P(t) dt = \frac{1}{4} \ln r - \frac{1}{4} \ln \Delta + \frac{3}{4} \ln \frac{A}{r} \quad (44)$$

we obtain a simpler form of eq. (40), namely

$$\frac{d^2 z(r)}{dr^2} + q(r) z(r) = 0 \quad (45)$$

with

$$\begin{aligned}q(r) &\equiv Q(r) - \frac{1}{2} \frac{dP(r)}{dr} - \frac{1}{4} P(r)^2 \\ &= -\frac{M(8r - 11M)}{4r^2(r - 2M)^2} + \frac{Mr^2}{A(r - 2M)} - \frac{2}{r^2} - \frac{m^2 r^4}{A\Delta} \\ &\quad + \frac{r^2(3r^4 - A)}{A^2} - \frac{\lambda_2 M^2 r^2 \Delta}{(n^2 + m^2 \omega^2) A^2 (r - 2M)^2} \\ &\quad + \frac{a^2}{A^2} (2r^4 - 2a^2 Mr - 3a^2 M^2) \\ &\quad - \frac{a^2 M}{rA(r - 2M)} (2r - 3M).\end{aligned}\quad (46)$$

Eq. (45) is reminiscent of the equation of a harmonic oscillator. Obviously, we do not plan to solve the general form of this equation, since after all we are interested only in what happens around our reference radius  $r_0$ . We will thus consider next particular limiting cases.

## 4 THE SCHWARZSCHILD CASE

### 4.1 Un-magnetized

We first consider the un-magnetized non-rotating case with  $a = \omega = B^2 = 0$ . In this case  $A = r^4$ ,  $\Delta = r^2 - 2Mr$ , and eq. (35) simplifies considerably while eq. (38) becomes

$$n^2 = \frac{m^2 a^r (1 - c_s^4) \mathcal{D}\{\rho\}}{r^2 \mathcal{D}\{(\rho + p)(\delta v^r)_{,r}\} / \delta v^r} \quad (47)$$

Taking into account the above considerations, eq. (45) admits the general solution

$$\begin{aligned}z(r) &= c_2 r^{(1/4)} \sqrt{(r - 2M)} P_{\xi-1/2}^{3/2} \left(\sqrt{\frac{r}{2M}}\right) \\ &\quad + c_1 r^{(1/4)} \sqrt{(r - 2M)} Q_{\xi-1/2}^{3/2} \left(\sqrt{\frac{r}{2M}}\right)\end{aligned}\quad (48)$$

where  $c_1, c_2$  are arbitrary constants,  $\xi \equiv \sqrt{1 + 4m^2}$ , and  $P_{\xi-1/2}^{3/2}(\sqrt{\frac{r}{2M}})$  and  $Q_{\xi-1/2}^{3/2}(\sqrt{\frac{r}{2M}})$  are the Legendre associate functions of first and second order respectively. In the limit  $M \rightarrow 0$  these functions behave as

$$\begin{aligned}P_\nu^\mu(z) &\sim \frac{\Gamma(\nu + 1/2)}{\sqrt{\pi} \Gamma(\nu + \mu + 1)} (2z)^\nu, \\ Q_\nu^\mu(z) &\sim \frac{\sqrt{\pi}}{2^{\nu+1} \Gamma(\nu + 3/2) z^{\nu+1}}\end{aligned}\quad (49)$$

(Oliver 1974), where  $z = \sqrt{\frac{r}{2M}}$ ,  $\nu = \xi - \frac{1}{2}$  and  $\mu = \frac{3}{2}$ . Furthermore, we define

$$\begin{aligned}\delta v_{(1)}^r(r) &= c_1 F(r) P_{\xi-1/2}^{3/2} \left(\sqrt{\frac{r}{2M}}\right) e^{nt+im\phi} \\ &\sim c_1 A_1 \left(1 - \frac{2M}{r}\right)^{3/4} r^{\xi/2-3/2} e^{nt+im\phi}, \quad \text{for } r < r_0 \\ \delta v_{(2)}^r(r) &= c_2 F(r) Q_{\xi-1/2}^{3/2} \left(\sqrt{\frac{r}{2M}}\right) e^{nt+im\phi} \\ &\sim c_2 A_2 \left(1 - \frac{2M}{r}\right)^{3/4} r^{-\xi/2-3/2} e^{nt+im\phi}, \quad \text{for } r > r_0\end{aligned}\quad (50)$$

where  $F(r) \equiv \left[\frac{2^{1/4}(r-2M)^{3/4}}{r^2}\right]$  and

$$\begin{aligned}A_1 &\equiv \frac{2^{\xi/2-1/4}}{\sqrt{\pi} M^{(\xi/2-1/4)} \xi (1 + \xi)} = \text{const.} \\ A_2 &\equiv \frac{\sqrt{\pi} M^{(\xi/2+1/4)}}{2^{\xi/2+1/4} \xi \Gamma(\xi)} = \text{const.}\end{aligned}\quad (51)$$

Putting everything back into eq. (47), after long calculations, we obtain the simple result

$$n^2 = \frac{m^2 \left(\frac{a^r}{r}\right) (1 - \frac{2M}{r}) (1 - c_s^4) \mathcal{D}\{\rho\}}{\xi(\rho + p) \left(1 - \frac{2M}{r}\right) - \frac{1}{2} \left(1 + \frac{M}{r}\right) \mathcal{D}\{\rho\}}. \quad (52)$$

Notice that all quantities that appear in the above equation imply their averages across the interface  $r = r_0$  (e.g.  $\rho \equiv (\rho_{(1)} + \rho_{(2)})/2$ , etc). For  $M = 0$ , eq. (52) reduces to eq. (51) in Chap. X of Chandrasekhar (1961).

### 4.2 Magnetized

The ‘force-balance’ eq. (35) becomes very complicated in the general magnetized case. In what follows, we will consider the general form of the stability criterion (eq. 38), but will at the same time adopt the expressions for  $\delta v^r$  across the interface that we obtained in the unmagnetized case. Under this approximation eq. (38) yields

$$\begin{aligned}n^2 &= m^2 \left(\frac{a^r}{r}\right) (1 - \frac{2M}{r}) \{ (1 - c_s^4) \mathcal{D}\{\rho\} - (1 + 3c_s^2) \mathcal{D}\{\frac{B^2}{8\pi}\} \} \\ &\quad / [\xi(\rho + p + \frac{B^2}{4\pi}) \left(1 - \frac{2M}{r}\right) - \frac{1}{2} \left(1 + \frac{M}{r}\right) \mathcal{D}\{\rho + p + \frac{B^2}{4\pi}\}]\end{aligned}\quad (53)$$

The denominator of the above equation is always positive, thus the sign of  $n^2$  is dictated by the sign of the numerator. Thus, in the limit  $c_s \rightarrow 0$ , the criterion for instability in the magnetized Schwarzschild case becomes

$$\mathcal{D}\{\rho\} - \mathcal{D}\{\frac{B^2}{8\pi}\} > 0, \quad (54)$$

which is the same as eq. (234) in Chap. X of Chandrasekhar (1961) obtained in the Newtonian limit.

## 5 THE KERR CASE

In Sec.4, we have examined the Rayleigh-Taylor instability in the presence of a dynamically significant magnetic field in a Schwarzschild space-time using the approximation that the solutions for  $\delta v^r$  inside and outside the interface  $r = r_0$  are those obtained in the un-magnetized Schwarzschild case. We will apply a similar approximation in the Kerr case. The ‘force-balance’ equation (eq. 86) now becomes complex

and results in two independent equations on the interface (eqs. 91 and 92). In what follows, we will consider only the first equation,  $\Lambda_1 = (N_1)_{,r}$ , since the second equation (93) yields a similar stability criterion. The expressions for  $\Lambda_1$  and  $N_1$  are given in eqs. (87) and (88) in the Appendix B.

As we pointed out above, we will proceed using the solutions of eq. (45) with  $\lambda_2 = 0$  as in the un-magnetized Schwarzschild case, only now  $a \neq 0$ . Because of its complexity, eq. (46) is still rather difficult to be solved analytically. However, if we only consider slowly rotating Kerr black holes with relatively small  $a$ , we can expand (46) in powers of  $a$  and keep terms up to  $a^2$ . Next, we expand the coefficient of  $a^2$  in powers of  $1/r$ , and keep only terms up to  $1/r$  and  $1/(r - 2M)$ . In this case, eq. (46) becomes

$$\frac{d^2 z(r)}{dr} = -[q_S(r) + a^2 q_K(r)]z(r) \quad (55)$$

where  $q_S(r)$ ,  $q_K(r)$  correspond to the Schwarzschild and Kerr space-times, respectively and their explicit forms are

$$\begin{aligned} q_S(r) &\equiv \frac{3M^2 - 4Mr - 4m^2(r^2 - 2Mr)}{4r^2(r - 2M)^2} \\ q_K(r) &\equiv \frac{8m^4 - 18m^2 + 9}{32m^2 M^2(r - 2M)} - \frac{8m^4 - 18m^2 + 9}{32m^2 M^3 r} \end{aligned} \quad (56)$$

Eq. (55) admits two general solutions

$$\begin{aligned} z_1(r) &= c_1 r^{1/4} \left(1 - \frac{2M}{r}\right)^{1/2} P_{\xi_K - 1/2}^{3/2} \left(\sqrt{\frac{r}{2M}}\right) \\ z_2(r) &= c_2 r^{1/4} \left(1 - \frac{2M}{r}\right)^{1/2} Q_{\xi_K - 1/2}^{3/2} \left(\sqrt{\frac{r}{2M}}\right) \end{aligned} \quad (57)$$

where  $\xi_K = \sqrt{(1 + 4m^2) - \frac{a^2}{4M^2 m^2}(m^2 - \frac{3}{4})(m^2 - \frac{3}{2})}$ .

Observe, that the solutions (57) differ from those of eqs. (48) only in the indices. Namely, in the Schwarzschild case, index  $\xi_K$  becomes equal to  $\xi$ . All the other factors in eqs. (57) are the same as in eq. (48). Thus, following the computations of subsection (4.1) and keeping terms only up to second order in  $a$  we end up with the criterion

$$\begin{aligned} n^2 &= -m^2 \omega^2 \\ &+ m^2 \left(\frac{M}{r^3}\right) \frac{r^6 \Delta}{A^2} [(1 - c_s^4) \mathcal{D}\{\rho\} - (1 + 3c_s^2) \mathcal{D}\{\frac{B^2}{8\pi}\}] \\ &/ \{ [\frac{r - 2M}{2\Delta A} [(6r^3 - 4Mr^2)a^2 + (4r - 9M)r^4] \\ &+ \frac{4M}{r} - \frac{5}{2}] \mathcal{D}\{\rho + p + \frac{B^2}{4\pi}\} \\ &+ \xi_K (1 - \frac{2M}{r})(\rho + p + \frac{B^2}{4\pi}) \} \end{aligned} \quad (58)$$

One can easily verify that when  $a^2 = 0$ , eq. (58) reduces to eq. (53).

As before, it is easy to show that the denominator in the r.h.s. of eq. (58) is always positive, and thus the stability criterion depends on the sign of the numerator. However, the new element here is that the Kerr geometry introduces a new term in the r.h.s., namely  $-m^2 \omega^2$  which softens the instability criterion. Thus, a configuration which would have been unstable in a non-rotating space-time, may now become stable. In other words, *the rotation of the space-time works in a direction that reduces the Rayleigh-Taylor instability*. As we will see next, this unexpected result has very important astrophysical applications.

## 6 SUMMARY AND DISCUSSION

Our goal has been to obtain a criterion for the onset of the magnetic Rayleigh-Taylor instability in curved space time. In order to achieve this goal, we made the following simplifying idealized assumptions:

- (i) We considered a disk configuration stationary with respect to ZAMOs (i.e. with velocity given by eq. 13).
- (ii) We investigated only what happens on the equatorial plane  $\theta = \pi/2$ , and in particular in the vicinity of some interface at radius  $r = r_0$ .
- (iii) We assumed dipolar symmetry in the magnetic field. The latter resulted in  $B^i = (0, B^\theta, 0)$ , and  $J^\mu = (0, 0, 0, J^\phi)$ .
- (iv) We assumed ideal MHD conditions in the form of eq. (10).
- (v) In order to make further progress, we assumed that  $\tilde{\nabla} \cdot \delta v = 0$ , and that  $\rho$ ,  $p$ , and  $B$  are uniform through the disk, with the exception of a jump in their values at some interface  $r_0$ .

Under the above conditions, we perturbed all physical quantities appearing in eqs. (3)-(9) to first order, we obtained the zero and first order equations (eqs. 73 and 74-85 respectively) under the Cowling approximation  $\delta g_{\mu\nu} = 0$ , and ended up with eq. (86) in the complex plane. The real part of that equation, eq. (35), is used to obtain both the dependence of the unknown function  $\delta v^r$  on the radial coordinate  $r$  away from the interface  $r = r_0$ , and the stability criterion at the interface itself. Notice that  $\delta v^r$  and  $(p + \frac{B^2}{8\pi})$  are continuous across the interface, but  $\rho$ ,  $B^2$  and  $(\delta v^r)_{,r}$  in general are not.

Eq. (52) expresses the stability criterion in the un-magnetized Schwarzschild space-time. To obtain the criterion in the magnetized Schwarzschild case, eq. (53), we used the solution for  $\delta v^r$  obtained in the unmagnetized case, eq. (45). Similarly, to obtain the criterion in the magnetized Kerr case, eq. (58), we used the solution for  $\delta v^r$  obtained in the unmagnetized case, eq. (55).

### 6.1 Astrophysical Implications

Let us here obtain a crude estimate of the maximum value of the magnetic field for which the disk-field configuration is stable. This is roughly also the maximum value of the magnetic field that can be held inside the inner edge of the disk at some radius  $r_0$  around the innermost stable orbit (ISCO) of a spinning black hole. In the limit of small  $a^2$ , negligible magnetic and gas pressure compared to the rest mass energy density  $\rho$ ,<sup>2</sup> and assuming a continuous matter distribution  $\mathcal{D}\{\rho\} = 0$  through the interface, eq. (58) yields the stability criterion (in real units)

$$-\mathcal{D}\{\frac{B^2}{8\pi}\} \lesssim \frac{\omega^2}{\Omega_K^2} \rho \approx \left(\frac{r_0}{r_S}\right)^{-3} \left(\frac{a}{M}\right)^2 \rho \quad (59)$$

<sup>2</sup> A crude estimate of the rest mass energy density at the Eddington accretion rate is  $\rho \sim GMm_p/r_0^2 \sigma_T \sim 4 \times 10^{14} M_1^{-1} \text{ erg/cm}^3$ , where  $M_1$  is the mass of the black hole in solar mass units, and  $\sigma_T$  is the electron Thomson cross section. The magnetic field  $B$  must be well below its equipartition value of  $B_{\text{eq}} \sim 10^8 M_1^{-1/2} \text{ G}$  (eq. 2) for our assumption of negligible magnetic pressure to apply.

(factors of order unity have been dropped from this calculation).  $r_s = 2GM/c^2$  is the Schwarzschild radius, and  $\Omega_K^2 \equiv GM/r_0^3$ . We have considered here only the most unstable mode with  $m = 1$  with  $\xi_K \approx 2$ , and assumed that  $r_0 \sim 6GM/c^2$ . If we further assume for simplicity that  $\mathcal{D}\{B^2\} \approx -B^2$ , i.e. if we assume that most of the field is brought inside  $r_0$ , eq. (59) yields

$$\frac{B_{\max}^2}{8\pi} \sim \frac{GMM_{\text{disk}}}{4\pi r_0^4} \left(\frac{a}{M}\right)^2, \quad (60)$$

which differs from the result of eq. (2) by a factor of order  $(a/M)^2$ ! We have assumed here a thick disk with mass  $M_{\text{disk}} \sim 4\pi r_0^3 \rho/c^2$ , and, as before, factors of order unity have been dropped from this order of magnitude estimate. The calculation may be crude, but leads to an important result, namely that even a small amount of poloidal magnetic field held inside the inner edge of an astrophysical accretion disk is unstable to the magnetic Rayleigh-Taylor instability, *unless* the central black hole is spinning.

One implication of this result is that non-rotating Magnetically Arrested Disks cannot exist around non-rotating black holes. MADs were first obtained in 2D general relativistic simulations where the Rayleigh-Taylor instability is obviously absent (e.g. Tchekhovskoy et al. 2010). MADs have also been obtained in 3D non-relativistic MHD simulations (e.g. Igumenshchev et al. 2003; Narayan et al. 2003) where rotation may play an important role in stabilizing the innermost disk against the Rayleigh-Taylor instability. Notice that Tchekhovskoy et al. (2012) sampled the full range of  $a/M$  and didn't observe the decrease in the average flux accumulated through the black hole horizon for low black hole spins implied by our present results<sup>3</sup>. We can only speculate that this is due to accretion: magnetic flux is advected inwards and at the same time escapes due to Rayleigh-Taylor instability, thus, on average, the amount of accumulated magnetic flux is non-zero. We may be able to account for the effect of accretion in a future publication.

We conclude by emphasizing that the magnetic Rayleigh-Taylor instability has serious implications for the origin of astrophysical jets and their associated radio emission. It is generally considered that some amount of the magnetic flux that is held by the accretion disk threads the horizon of the central black hole. As a result, relativistic jet outflows are expected both from the vicinity of the black hole and the inner accretion disk, therefore, it is hard to separate their respective contributions to the total jet power (Christodoulou et al. 2016, submitted). Observations tend to support such a combined structure with the corresponding models referred to as “spine - sheath” (Ghisellini et al. 2005), with both components contributing to the jet power. According to Blandford & Znajek (1977), if the central black hole is spinning, a highly relativistic black hole jet is generated which extracts power

$$P_{\text{BZ}} \sim \frac{1}{c} B_{\text{BH}}^2 r_{\text{BH}}^4 \omega_{\text{BH}}^2 \quad (61)$$

(Blandford & Znajek 1977; Tchekhovskoy et al. 2010; Nathanail & Contopoulos 2014). Here,  $B_{\text{BH}}$  is the value of

the magnetic field that threads the black hole horizon (this is roughly the same as the value of the magnetic field that is held inside the inner edge of the disk),  $r_{\text{BH}}$  is the radius of the horizon, and  $\omega_{\text{BH}}$  is the black hole angular frequency. It is, therefore, imperative to understand how the magnetic Rayleigh-Taylor instability limits the maximum possible accumulated magnetic flux. We thus plan to continue our investigation in the presence of accretion and rotation.

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<sup>3</sup> In fact, they observed a slight decrease at high black hole spins which we believe may be associated to the shrinking of the black hole horizon with spin.

## APPENDIX A: USEFUL EXPRESSIONS

Below we have collected some useful expressions

$$\begin{aligned} \frac{1}{\alpha}[\tilde{\nabla} \cdot (\alpha \tilde{J})] &= \tilde{\nabla} \cdot \tilde{J} + \tilde{J} \cdot \frac{\tilde{\nabla} \alpha}{\alpha} = \tilde{\nabla} \cdot \tilde{J} + \tilde{J} \cdot \tilde{a} \\ &= [J_{,i}^i + \Gamma_{il}^i J^l] + \gamma_{ij} a^i J^j, \end{aligned} \quad (62)$$

where  $\tilde{a} = \tilde{\nabla} \alpha / \alpha$  is the acceleration,

$$\begin{aligned} \frac{1}{\alpha^2}[\tilde{\nabla} \cdot (\alpha^2 \tilde{S})] &= \tilde{\nabla} \cdot \tilde{S} + \tilde{S} \cdot \frac{\tilde{\nabla} \alpha^2}{\alpha^2} = \tilde{\nabla} \cdot \tilde{S} + 2\tilde{J} \cdot \tilde{a} \\ &= [S_{,i}^i + \Gamma_{il}^i S^l] + \gamma_{ij} a^i S^j \\ \frac{1}{\alpha}[\tilde{\nabla} \cdot (\alpha \tilde{W})] &= \tilde{\nabla} \cdot \tilde{W} + \tilde{W} \cdot \frac{\tilde{\nabla} \alpha}{\alpha} = \tilde{\nabla} \cdot \tilde{W} + \tilde{W} \cdot \tilde{a} \\ &= [W_{,i}^{ki} + \Gamma_{il}^i W^{kl}] + \gamma_{ij} a^i W^{kj} \end{aligned} \quad (63)$$

$$\begin{aligned} \frac{1}{\alpha}[\tilde{\nabla} \times (\alpha \tilde{B})] &= \tilde{\nabla} \times \tilde{B} - \tilde{B} \times \frac{\tilde{\nabla} \alpha}{\alpha} = \tilde{\nabla} \times \tilde{B} - \tilde{B} \times \tilde{a} \\ \frac{1}{\alpha}[\tilde{\nabla} \times (\alpha \tilde{E})] &= \tilde{\nabla} \times \tilde{E} - \tilde{E} \times \frac{\tilde{\nabla} \alpha}{\alpha} = \tilde{\nabla} \times \tilde{E} - \tilde{E} \times \tilde{a} \end{aligned} \quad (64)$$

In Kerr space-time

$$\frac{1}{\alpha}[\rho_{,\mu}(\alpha U^\mu + \beta^\mu) - \gamma^{ij} \beta_i \rho_{,j}] = \frac{1}{\alpha}[\rho_{,0} + \omega \rho_{,\phi}] \quad (65)$$

## APPENDIX B: GRMD EQUATIONS

Below we summarize the general relativistic MHD equations.

### 6.2 Zeroth Order Equations

$$\begin{aligned} \rho_e &= 0 \\ B_{,\theta}^\theta + \frac{\cot \theta}{\Sigma} [r^2 + a^2 M^2 (1 - 3 \sin^2 \theta)] B^\theta &= 0 \quad B_{,\phi}^\theta = 0 \\ J^r &= -\frac{\alpha}{4\pi \Sigma \sin \theta} B_{\theta,\phi}, \quad J^\theta = 0 \\ J^\phi &= \frac{\alpha}{4\pi \Sigma \sin \theta} [B_{\theta,r} - B_{r,\theta} + a_r B_\theta] \\ J_{,r}^r + J_{,\phi}^\phi + \left(\frac{\Sigma}{\Delta} a^r + \frac{2r}{\Sigma}\right) J^r &= 0 \\ \rho_{,\phi} = 0 &\Rightarrow \rho = \rho(r, \theta) \\ (\rho + p) a^r + \left(\frac{\Delta}{\Sigma}\right) p_{,r} &= -\frac{\alpha}{\Sigma \sin \theta} J_\phi B_\theta \\ (\rho + p) a^\theta + \left(\frac{1}{\Sigma}\right) p_{,\theta} &= 0 \\ \frac{A}{\Sigma \sin^2 \theta} p_{,\phi} &= \frac{\alpha}{\Sigma \sin \theta} J_r B_\theta. \end{aligned} \quad (66)$$

### 6.3 First Order Equations

$$\begin{aligned} \delta E^r &= \frac{\alpha}{\Sigma \sin \theta} B_\theta \delta v_\phi, \quad \delta E^\theta = 0, \quad \delta E^\phi = -\frac{\alpha}{\Sigma \sin \theta} B_\theta \delta v_r \\ \delta E_{,r}^r + \delta E_{,\phi}^\phi + \frac{1}{\Sigma} [2r \delta E^r + (\cot \theta \Sigma - 2a^2 M^2 \sin \theta \cos \theta) \delta E^\theta] &= 4\pi \delta \rho_e \\ \delta B_{,r}^r + \delta B_{,\theta}^\theta + \delta B_{,\phi}^\phi &+ \frac{1}{\Sigma} [2r \delta B^r + (\cot \theta \Sigma - 2a^2 M^2 \sin \theta \cos \theta) \delta B^\theta] = 0 \end{aligned} \quad (67)$$

$$\begin{aligned} 4\pi \delta J^r &= \frac{\alpha}{\Sigma \sin \theta} [\delta B_{\phi,\theta} - \delta B_{\theta,\phi} + a_\theta \delta B_\phi] \\ &- \frac{1}{\alpha} [\delta E_{,t}^r + \omega \delta E_{,\phi}^r] - \frac{\delta E^\phi}{\alpha} (\Gamma_{t\phi}^r + \omega \Gamma_{\phi\phi}^r) + \gamma_{\phi\phi} \sigma^{r\phi} \delta E^\phi \\ 4\pi \delta J^\theta &= \frac{\alpha}{\Sigma \sin \theta} [\delta B_{r,\phi} \\ &- \delta B_{\phi,r} - a_r \delta B_\phi] - \frac{\delta E^\phi}{\alpha} (\Gamma_{t\phi}^\theta + \omega \Gamma_{\phi\phi}^\theta) \\ 4\pi \delta J^\phi &= \frac{\alpha}{\Sigma \sin \theta} [\delta B_{\theta,r} - \delta B_{r,\theta} + a_r \delta B_\theta - a_\theta \delta B_r] \\ &- \frac{1}{\alpha} [\delta E_{,t}^\phi + \omega \delta E_{,\phi}^\phi] - \frac{\delta E^r}{\alpha} (\Gamma_{tr}^\phi + \omega \Gamma_{\phi r}^\phi - \omega a_r) \\ &+ \gamma_{rr} \sigma^{r\phi} \delta E^r \\ &\frac{1}{\alpha} (\delta B_{,t}^r + \omega \delta B_{,\phi}^r) + (\Gamma_{t\phi}^r + \omega \Gamma_{\phi\phi}^r) \frac{\delta B^\phi}{\alpha} - \gamma_{\phi\phi} \sigma^{r\phi} \delta B^\phi \\ &= \frac{\alpha}{\Sigma \sin \theta} (\delta E_{\theta,\phi} - \delta E_{\phi,\theta} - a_\theta \delta E_\phi) \\ &\frac{1}{\alpha} (\delta B_{,\theta}^\theta + \omega \delta B_{,\phi}^\theta) + (\Gamma_{t\phi}^\theta + \omega \Gamma_{\phi\phi}^\theta) \frac{\delta B^\phi}{\alpha} - \gamma_{\phi\phi} \sigma^{\theta\phi} \delta B^\phi \\ &= \frac{\alpha}{\Sigma \sin \theta} (\delta E_{\phi,r} - \delta E_{r,\phi} + a_r \delta E_\phi) \\ &\frac{1}{\alpha} (\delta B_{,t}^\phi + \omega \delta B_{,\phi}^\phi) + (\Gamma_{tr}^\phi + \omega \Gamma_{\phi r}^\phi - \omega a_r) \frac{\delta B^r}{\alpha} \\ &+ (\Gamma_{t\theta}^\phi + \omega \Gamma_{\phi\theta}^\phi - \omega a_\theta) \frac{\delta B^\theta}{\alpha} - \gamma_{rr} \sigma^{r\phi} \delta B^r - \gamma_{\theta\theta} \sigma^{\theta\phi} \delta v^\theta \\ &= \frac{\alpha}{\Sigma \sin \theta} (\delta E_{r,\theta} - \delta E_{\theta,r} - a_r \delta E_\theta + a_\theta \delta E_r) \end{aligned} \quad (68)$$

$$\begin{aligned} \frac{1}{\alpha} (\delta \rho_{e,t} + \omega \delta \rho_{e,\phi}) &= -\left(\frac{\Sigma}{\Delta}\right) a^r \delta J^r - \Sigma a^\theta \delta J^\theta \\ &+ \delta J_{,r}^r + \delta J_{,\theta}^\theta + \delta J_{,\phi}^\phi \\ &+ \frac{1}{\Sigma} [2r \delta J^r + (\Sigma \cot \theta - 2a^2 M^2 \sin \theta \cos \theta) \delta J^\theta] \\ &\frac{1}{\alpha} (\delta \rho_{,t} + \omega \delta \rho_{,\phi}) + 2(\rho + p) [a_r \delta v^r + a_\theta \delta v^\theta] \\ &+ [\delta v^r (\rho + p)_{,r} + \delta v^\theta (\rho + p)_{,\theta} + \delta v^\phi (\rho + p)_{,\phi}] \\ &= -\left(\frac{\Sigma}{\Delta}\right) J^r \delta E^r - \left(\frac{A \sin^2 \theta}{\Sigma}\right) J^\phi \delta E^\phi \end{aligned} \quad (70)$$

$$\begin{aligned} &\frac{(\rho + p)}{\alpha} (\delta v_{,t}^r + \omega \delta v_{,\phi}^r) + \frac{(\rho + p)}{\alpha} (\Gamma_{t\phi}^r + \omega \Gamma_{\phi\phi}^r) \delta v^\phi \\ &+ a^r (\delta \rho + \delta p) + \left(\frac{\Delta}{\Sigma}\right) \delta p_{,r} + (\rho + p) \gamma_{\phi\phi} \sigma^{r\phi} \delta v^\phi \\ &= -\frac{\alpha}{\Sigma \sin \theta} (\delta J_\phi B_\theta - \delta B_\phi J_\theta + J_\phi \delta B_\theta) \\ &\frac{(\rho + p)}{\alpha} (\delta v_{,t}^\theta + \omega \delta v_{,\phi}^\theta) + \frac{(\rho + p)}{\alpha} (\Gamma_{t\phi}^\theta + \omega \Gamma_{\phi\phi}^\theta) \delta v^\phi \\ &+ a^\theta r (\delta \rho + \delta p) + \left(\frac{1}{\Sigma}\right) \delta p_{,\theta} + (\rho + p) \gamma_{\phi\phi} \sigma^{\theta\phi} \delta v^\phi \\ &= -\frac{\alpha}{\Sigma \sin \theta} (J_r \delta B_\phi - \delta B_r J_\phi) \\ &\frac{(\rho + p)}{\alpha} (\delta v_{,t}^\phi + \omega \delta v_{,\phi}^\phi) + \frac{(\rho + p)}{\alpha} [(\Gamma_{tr}^\phi + \omega \Gamma_{\phi r}^\phi - \omega a_r) \delta v^r \\ &+ (\Gamma_{t\theta}^\phi + \omega \Gamma_{\phi\theta}^\phi - \omega a_\theta) \delta v^\theta] + \left(\frac{\Sigma}{A \sin^2 \theta}\right) \delta p_{,\phi} \\ &+ (\rho + p) (\gamma_{rr} \sigma^{r\phi} \delta v^r + \gamma_{\theta\theta} \sigma^{\theta\phi} \delta v^\theta) \\ &= -\frac{\alpha}{\Sigma \sin \theta} (\delta B_r J_\theta - \delta J_r B_\theta - J_r \delta B_\theta) \end{aligned} \quad (71)$$



(72)

#### 6.4 The equatorial plane

On the equatorial plane the zero order eqs. (66), with contravariant and some of them with covariant indices needed for our work, reads

$$\begin{aligned}
\rho_e &= 0 \\
B_{,\theta}^\theta &= 0, \quad B_{,\phi}^\theta = 0 \\
J^r &= 0, \quad J^\theta = 0, \quad J_{,\phi}^\phi = 0 \\
J^\phi &= \frac{\alpha}{4\pi\Sigma}[B_{\theta,r} + a_r B_\theta] \\
J_r &= 0, \quad J_\theta = 0, \\
J_\phi &= \left(\frac{1}{4\pi}\right)\left(\frac{\Delta}{\alpha}\right)(B_{,r}^\theta + \frac{2}{r}B^\theta + a_r B^\theta), \\
\rho_{,\phi} &= 0 \Rightarrow \rho = \rho(r, \theta) \\
(\rho + p)a^r + \left(\frac{\Delta}{\Sigma}\right)p_{,r} &= -\frac{\alpha}{\Sigma}J_\phi B_\theta \\
(\rho + p)a^\theta + \left(\frac{1}{\Sigma}\right)p_{,\theta} &= 0 \\
p_{,\phi} &= \frac{\alpha}{A}J_r B_\theta.
\end{aligned} \tag{73}$$

The first order equations eqs. (67)-(72) simplifies considerably on the equatorial plane, where we have  $a^\theta = 0$ ,  $\sigma^{\theta\phi} = 0$ ,  $\Gamma_{t\phi}^\theta = \Gamma_{\phi\phi}^\theta = 0$ , and  $\Gamma_{\theta\theta}^\phi = \Gamma_{\theta\phi}^\phi = 0$ . Our anzatz (eq. 34), because of eqs. (29) becomes

$$\delta v_{,r}^r + \delta v_{,\theta}^\theta + \delta v_{,\phi}^\phi + \frac{2}{r}\delta v^r = 0 \tag{74}$$

which in turn, gives

$$-im\delta v^\phi = \frac{1}{r^2}(r^2\delta v^r)_{,r} \equiv \chi. \tag{75}$$

Furthermore, with the aid of eqs. (32), the system of first order equations, eqs. (67)-(72) and (75), yield

$$\begin{aligned}
\delta B^r &= \delta B^\phi = 0 \\
\delta B^\theta &= -(n - im\omega)\alpha\Xi_1\delta v^r
\end{aligned} \tag{76}$$

where  $\Xi_1 = \frac{B_{,r}^\theta}{n^2 + m^2\omega^2}$ .

Eqs. (67) yield

$$\begin{aligned}
\delta E^r &= \frac{\alpha}{r^2}B_\theta\delta v_\phi, \quad \delta E^\theta = 0, \quad \delta E^\phi = -\frac{\alpha}{r^2}B_\theta\delta v_r \\
\delta E_r &= \frac{r^2}{\alpha}B^\theta\delta v^\phi, \quad \delta E_\theta = 0, \quad \delta E_\phi = -\frac{r^2}{\alpha}B^\theta\delta v^r \\
\delta E_{,r}^r + \delta E_{,\phi}^\phi + \frac{2}{r}\delta E^r &= 4\pi\delta\rho_e.
\end{aligned} \tag{77}$$

Eq. (71) gives

$$\begin{aligned}
\delta\rho &= -\frac{\alpha(n - im\omega)}{n^2 + m^2\omega^2}[(1 - c_s^2)\rho_{,r} \\
&\quad - \frac{3}{4\pi}(B_\theta B_{,r}^\theta + \frac{2r}{\Sigma}B^2 + a_r B^2)]\delta v^r
\end{aligned} \tag{78}$$

Eqs. (68) become

$$\begin{aligned}
4\pi\delta J^r &= -\frac{\alpha}{\Sigma}\delta B_{\theta,\phi} \\
-\frac{1}{\alpha}[\delta E_{,t}^r + \omega\delta E_{,\phi}^r] - \frac{\delta E^\phi}{\alpha}(\Gamma_{t\phi}^r + \omega\Gamma_{\phi\phi}^r - \alpha\gamma_{\phi\phi}\sigma^{r\phi}), \\
4\pi\delta J^\theta &= 0,
\end{aligned}$$

$$\begin{aligned}
4\pi\delta J^\phi &= \frac{\alpha}{\Sigma}[\delta B_{\theta,r} + a_r\delta B_\theta] \\
-\frac{1}{\alpha}[\delta E_{,t}^\phi + \omega\delta E_{,\phi}^\phi] - \frac{\delta E^r}{\alpha}(\Gamma_{tr}^\phi + \omega\Gamma_{\phi r}^\phi - \omega a_r - \alpha\gamma_{rr}\sigma^{\phi r}).
\end{aligned} \tag{79}$$

Since we need the covariant components of Eqs(79) we find

$$\begin{aligned}
4\pi\delta J_r &= -\left(\frac{\Sigma^2}{\alpha A}\right)\delta B_{,\phi}^\theta - \left(\frac{\Sigma^2}{\alpha A}\right)\sigma_{r\phi}B^\theta\delta v^r - D_\tau\delta E_r, \\
4\pi\delta J_\theta &= 0, \\
4\pi\delta J_\phi &= \left(\frac{\Delta}{\alpha}\right)[\delta B_{,r}^\theta + \frac{2r}{\Sigma}\delta B^\theta] \\
&\quad + \left(\frac{\Sigma}{\alpha}\right)d^r B^\theta - \left(\frac{\Delta}{\alpha}\right)\sigma_{\phi r}B^\theta\delta v^\phi - D_\tau\delta E_\phi
\end{aligned} \tag{80}$$

Further, eqs. (72) reduce to the system

$$\begin{aligned}
-\left(\frac{\Delta}{r^2}\right)(\delta p)_{,r} &= \frac{(\rho + p)}{\alpha}(n + im\omega)\delta v^r \\
&\quad + \frac{(\rho + p)}{\alpha}[G_2(r) + \alpha\left(\frac{A}{\Sigma}\right)\sigma^{r\phi}]\left(\frac{i}{m}\chi\right) \\
&\quad - a^r\alpha(1 + c_s^2)\left[\frac{n - im\omega}{n^2 + m^2\omega^2}\right][(1 - c_s^2)\rho_{,r} \\
&\quad - \frac{3}{4\pi}(B_\theta B_{,r}^\theta + \frac{2}{r}B^2 + a_r B^2)]\delta v^r \\
&\quad + \frac{\alpha}{r^2}[\delta J_\phi B_\theta + J_\phi\delta B_\theta]
\end{aligned} \tag{81}$$

and

$$\begin{aligned}
-\left(\frac{r^2}{A}\right)(\delta p)_{,\phi} &= \frac{(\rho + p)}{\alpha}(n + im\omega)\left(\frac{i}{im}\chi\right) \\
&\quad + \frac{(\rho + p)}{\alpha}[G_1(r) + \alpha\left(\frac{\Sigma}{\Delta}\right)\sigma^{\phi r}]\delta v^r \\
&\quad - \left(\frac{\alpha}{r^2}\right)B_\theta\delta J_r
\end{aligned} \tag{82}$$

where

$$\begin{aligned}
4\pi B_\theta\delta J_r &= -\left(\frac{\Sigma^2}{\alpha A}\right)B_\theta\delta B_{,\phi}^\theta - \left(\frac{\Sigma^2}{\alpha A}\right)\sigma_{r\phi}B^2\delta v^r - B_\theta D_\tau\delta E_r, \\
4\pi[B_\theta\delta J_\phi + J_\phi\delta B_\theta] &= \left(\frac{\Sigma\Delta}{\alpha\Sigma}\right)[B_\theta\delta B_{,r}^\theta + B_{,r}^\theta\delta B_\theta \\
&\quad + \frac{4}{r}B^\theta\delta B_\theta + 2a_r B_\theta\delta B^\theta] + \left(\frac{\Sigma\Delta}{\alpha\Sigma}\right)\sigma_{r\phi}\left[\frac{i\chi}{m}\right]B^2 - B_\theta D_\tau\delta E_\phi,
\end{aligned} \tag{83}$$

and because of the  $D_\tau M^\beta \equiv M^\beta{}_{;\mu}U^\mu - U^\beta a_\mu M^\mu$ , which is the Fermi derivative,

$$\begin{aligned}
B_\theta D_\tau\delta E_r &= \left(\frac{\Sigma}{\alpha^2}\right)\left[\frac{i\chi}{m}\right](n + im\omega)B^2 + \left(\frac{\Sigma}{\alpha^2}\right)G_3(r)B^2\delta v^r, \\
B_\theta D_\tau\delta E_\phi &= -\left(\frac{\Sigma}{\alpha^2}\right)(n + im\omega)B^2\delta v^r - \left(\frac{\Sigma}{\alpha^2}\right)\left[\frac{i\chi}{m}\right]G_2(r)B^2,
\end{aligned} \tag{84}$$

We have defined above

$$\begin{aligned}
G_1(r) &\equiv \Gamma_{tr}^\phi + \omega\Gamma_{r\phi}^\phi - a_r\omega \\
&= \left(\frac{\omega}{2r}\right)\left[\frac{3r(r - 2M)^2 + a^2(r - 4M)}{(r - 2M)\Delta}\right] \\
&\equiv \left(\frac{\omega}{2r}\right)\tilde{G}_1(r), \\
G_2(r) &\equiv \Gamma_{t\phi}^r + \omega\Gamma_{\phi\phi}^r \\
&= -\left(\frac{\omega}{2r}\right)\left[\frac{(3r^2 + a^2)\Delta}{r^2}\right] \\
&\equiv -\left(\frac{\omega}{2r}\right)\tilde{G}_2(r), \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
G_3(r) &\equiv G_1(r) + a_r \omega \\
&= \left(\frac{\omega}{2r}\right) \left[ \frac{3r^2 - 4Mr + a^2}{\Delta} \right] \\
&\equiv \left(\frac{\omega}{2r}\right) \tilde{G}_3(r).
\end{aligned} \tag{85}$$

From eqs. (81) and (82), using eqs. (91)-(85), we find a complex equation of the form

$$\Lambda_1 + i\Lambda_2 = (N_1)_{,r} + i(N_2)_{,r} \tag{86}$$

where

$$\begin{aligned}
\Lambda_1 \equiv & n\left(\frac{1}{\alpha\Delta}\right)(\rho + p + \frac{B^2}{4\pi})(r^2\delta v^r) \\
& - \left(\frac{\alpha a^r}{\Delta}\right) \left[ \frac{n}{n^2 + m^2\omega^2} \right] [(1 - c_s^4)\rho_{,r} \\
& - \left(\frac{3}{4\pi}\right)(1 + c_s^2)(B_\theta B_{,r}^\theta + \frac{2}{r}B^2 + a_r B^2)](r^2\delta v^r) \\
& \left(\frac{2}{4\pi}\right) \left(\frac{1}{r} + a_r\right) (B^\theta \delta B_\theta)
\end{aligned} \tag{87}$$

$$\begin{aligned}
\Lambda_2 \equiv & (m\omega) \left(\frac{1}{\alpha\Delta}\right) (\rho + p + \frac{B^2}{4\pi})(r^2\delta v^r) \\
& + \frac{(\rho + p + \frac{B^2}{4\pi})}{\alpha\Delta} [G_2(r) + \left(\frac{\alpha A}{r^2}\right) \sigma^{r\phi}] \left(\frac{r^2}{m}\chi\right) \\
& + \left(\frac{\alpha a^r}{\Delta}\right) \left[ \frac{m\omega}{n^2 + m^2\omega^2} \right] [(1 - c_s^4)\rho_{,r} \\
& - \left(\frac{3}{4\pi}\right)(1 + c_s^2)(B_\theta B_{,r}^\theta + \frac{2}{r}B^2 + a_r B^2)](r^2\delta v^r) \\
& \left(\frac{2}{4\pi}\right) \left(\frac{1}{r} + a_r\right) (B^\theta \delta B_\theta)
\end{aligned} \tag{88}$$

$$\begin{aligned}
N_1 \equiv & \left(\frac{nr^2\chi}{m^2}\right) \left(\frac{1}{\alpha}\right) \left(\frac{A}{r^4}\right) (\rho + p + \frac{B^2}{4\pi}) \\
& + \left(\frac{1}{4\pi}\right) B_\theta \delta B^\theta
\end{aligned} \tag{89}$$

and

$$\begin{aligned}
N_2 \equiv & (m\omega) \left(\frac{1}{\alpha}\right) \left(\frac{A}{r^4}\right) \left(\frac{r^2\chi}{m^2}\right) (\rho + p + \frac{B^2}{4\pi}) \\
& - \left(\frac{1}{m}\right) \left(\frac{A}{r^4}\right) \frac{(\rho + p)}{\alpha} G_1(r) (r^2\delta v^r) \\
& - \left(\frac{1}{m}\right) \left(\frac{A}{r^2\Delta}\right) \sigma^{\phi r} (\rho + p + \frac{B^2}{4\pi}) (r^2\delta v^r) \\
& - \left(\frac{1}{m}\right) \left(\frac{1}{\alpha}\right) \left(\frac{A}{r^4}\right) G_3(r) \left(\frac{B^2}{4\pi}\right) (r^2\delta v^r)
\end{aligned} \tag{90}$$

Obviously, from eq. (86) we have the following two equations

$$\Lambda_1 = (N_1)_{,r} \tag{91}$$

and

$$\Lambda_2 = (N_2)_{,r} \tag{92}$$

In the main text we consider only eq. (91) since eq. (92) does not give any new and significantly different results. Using eqs. (90) and (87), eq. (92) may be written as

$$\begin{aligned}
& -F(r) \left\{ \left(\frac{A}{r^4}\right) (\rho + p + \frac{B^2}{4\pi}) (r^2\delta v^r)_{,r} \right. \\
& - \left(\frac{A}{r^4}\right) \left[ \frac{\tilde{G}_1(r)}{2r} (\rho + p + \frac{B^2}{4\pi}) + a_r \frac{B^2}{4\pi} \right] (r^2\delta v^r) \\
& \left. - \lambda_\sigma \frac{r^2(3r^2 + a^2)}{2A} (\rho + p + \frac{B^2}{4\pi}) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left[ \left(\frac{A}{r^4}\right) (\rho + p + \frac{B^2}{4\pi}) (r^2\delta v^r)_{,r} \right]_{,r} \\
& - \left\{ \left(\frac{A}{r^4}\right) \left[ \frac{\tilde{G}_1(r)}{2r} (\rho + p + \frac{B^2}{4\pi}) + a_r \left(\frac{B^2}{4\pi}\right) \right] (r^2\delta v^r) \right. \\
& \left. - \lambda_\sigma \frac{r^2(3r^2 + a^2)}{2A} (\rho + p + \frac{B^2}{4\pi}) \right\}_{,r} \\
& = \frac{m^2}{\Delta} (r^2\delta v^r) (\rho + p + \frac{B^2}{4\pi}) - \left[ \frac{\tilde{G}_2(r)}{2r\Delta} \right. \\
& \left. + \lambda_\sigma \frac{r^2(3r^2 + a^2)}{2A} \right] (\rho + p + \frac{B^2}{4\pi}) (r^2\delta v^r)_{,r} \\
& + \left[ \frac{m^2}{n^2 + m^2\omega^2} \right] \left(\frac{\alpha^2 a^r}{\Delta}\right) [(1 - c_s^4)\rho_{,r} - \frac{1}{4\pi}(1 + 3c_s^2)B_\theta B_{,r}^\theta \\
& - \left(\frac{3}{4\pi}\right)(1 + c_s^2) \left(\frac{2}{r} + a_r\right) B^2 \\
& + \left(\frac{2\Delta}{ra^r}\right) \left(\frac{1}{4\pi}\right) (B_\theta \delta B^\theta)_{,r}] (r^2\delta v^r)
\end{aligned} \tag{93}$$

where  $\tilde{G}_1(r)$ ,  $\tilde{G}_2(r)$  are given by eq. (85)  $\lambda_\sigma$  is a constant that is related to the shear  $\sigma^{r\phi}$  term and  $F(r)$  is

$$F(r) \equiv \frac{1}{A\Delta} [3r^5 - 5Mr^4 + 4a^2r^3 + a^4r - 4M^2a^2r + Ma^4] \tag{94}$$